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Adaptative Remeshing for Viscous Incompressible Flows

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This paper presents an adaptive finite element procedure for solving viscous incompressible flows. The methodology is based on adaptive remeshing for steady-state problems. The Navier-Stokes equations for an incompressible fluid are solved in primitive variables by an Uzawa algorithm using a highly accurate element. The efficiency and convergence rate of the adaptive strategy are evaluated by solving problems with known analytical solutions. Finally, the methodology is applied to the flow over a backward-facing step and predictions are compared with experimental measurements. The use of the proposed adaptive procedure is shown to lead to improved accuracy of the finite element predictions.

Nomenclature

C = a constant
e = error in the solution
h = element size
$J(\cdot)$ = dissipation energy
n = number of elements in the mesh
p = pressure
S = step and fence height
U = velocity vector
u, v = velocity components
x, y = coordinates
∇ = gradient operator
δ = element size predicted by adaptation module
ϵ = strain rate tensor $= \frac{1}{2}(\nabla U + \nabla U^T)$
σ = stress tensor $= 2\mu\epsilon$
ξ = similarity variable in a boundary layer
η = relative error on a mesh
μ = absolute viscosity of the fluid
ν = kinematic viscosity of the fluid

Subscripts

av = average error, error in optimal mesh
ex = exact solution
h = smoothed approximation obtained from a discontinuous one
t = target
0 = freestream value

Introduction

OVER the past few years adaptive methods have stirred much interest in the engineering community. The first inroads were achieved in compressible flow applications because of the pressing need for accurate computation of shock waves.[1-6] Adaptive methods offer a means of tackling complex flow problems at a reasonable cost and of controlling the accuracy of numerical simulations. Although spectacular results were achieved for aerodynamic flow, very little has been accomplished for viscous incompressible flows.

This paper presents an adaptive remeshing finite element method for viscous incompressible flows. Such flows present special challenges for adaptive methods. Because of the elliptic nature of the Navier-Stokes equations a fully coupled approach is used. Moreover, the discretization of the equations requires some care if meaningful results are to be obtained.

The paper is organized as follows: the methodology section discusses mesh generation, the finite element algorithm, error estimation, and the adaptive strategy. The methodology is then validated by solving problems with known analytical solutions to clearly quantify improvements due to adaptivity. Finally, the method is applied to the flow over a backward-facing step for which experimental data is available.

Methodology

Generalities

There are several ways of achieving adaptivity: P methods vary the degree of the polynomial approximation depending on the solution,[2] R methods relocate grid points to improve accuracy,[4] and H methods proceed by grid enrichment or remeshing.[5-7]

A variant of an H method, called adaptive remeshing, has been retained. It provides the greatest control of mesh size and grading to better resolve the flow features. Furthermore, adaptive remeshing offers the possibility of transforming an existing incompressible finite element flow code into an adaptive solver.

In this method the problem is first solved on a grid fine enough to roughly capture the physics of the flow, such as boundary layers, whose location and extent are unknown a priori. The resulting solution is then analyzed to determine where more grid points are needed and an improved mesh is then generated. The process is repeated until the required level of accuracy is achieved.

Incompressibility Considerations

Remeshing offers an elegant and simple approach for overcoming some of the obstacles specific to incompressible viscous flows. The best proven finite element approximations can be selected based on their convergence and accuracy proper-

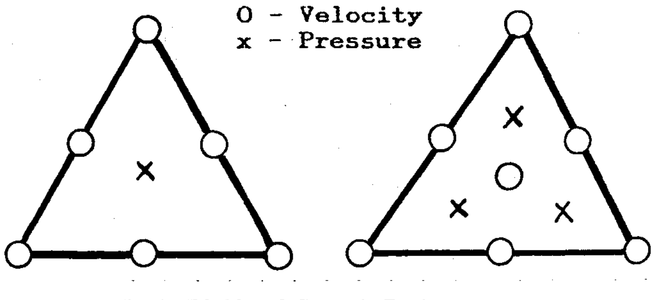

Fig. 1 P2-P0 and Crouzeix-Raviart elements.

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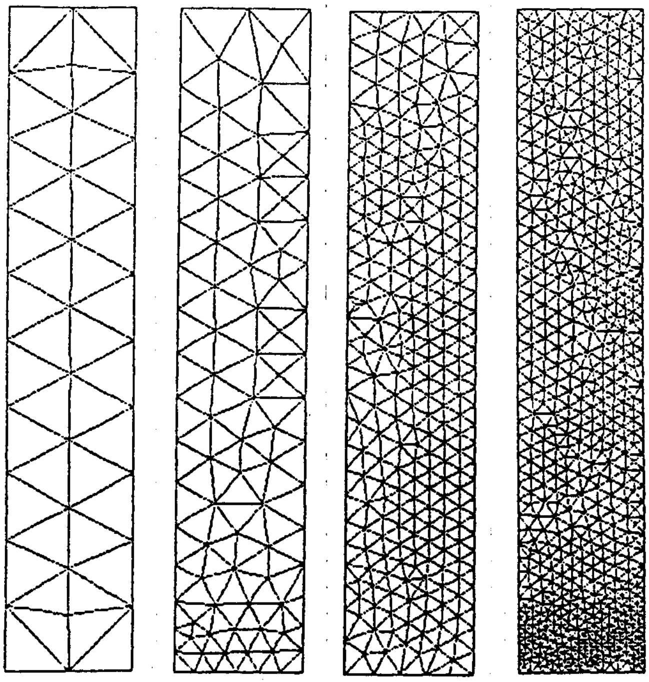

Fig. 2 One-dimensional boundary layer: meshes P2-P0 element.

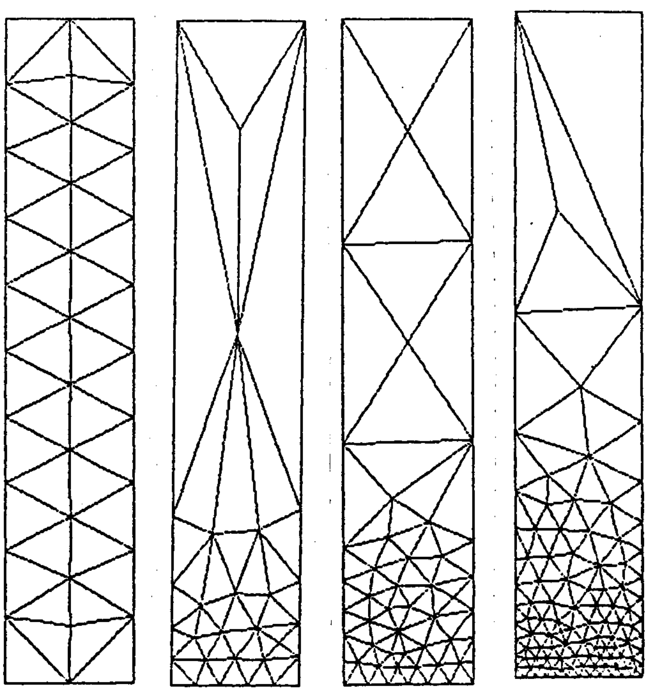

Fig. 3 One-dimensional boundary layer: meshes Crouzeix-Raviart element.

ties.[8] This circumvents the problems associated with P methods and eliminates the "hanging-node problem" encountered in h refinement.

Remeshing is implemented with the two finite element approximations shown in Fig. 1; both elements satisfy the so-called LBB compatibility condition between the velocity and pressure discretizations. The first element uses quadratic polynomials for the velocity and a piecewise constant approximation for the pressure. This element is very economical but not very accurate.

The second element, called the Crouzeix-Raviart, uses a quadratic velocity field approximation enriched with a bubble function[8] and discontinuous piecewise linear pressure on each

Table 1 One-dimensional boundary layer, P2-P0 element constant reduction of error

Mesh	Grid points	$\|U\|$	$\|e_{ex}\|$	Relative error, %
0	109	0.316228	0.03487	11.0
1	317	0.316228	0.01604	5.1
2	1005	0.316226	0.00900	2.8
3	2415	0.316226	0.00560	1.8

Table 2 One-dimensional boundary layer, P2-P0 element target: 1% relative error

Mesh	Grid points	$\|U\|$	$\|e_{ex}\|$	Relative error, %
0	109	0.316228	0.02141	6.8
1	1917	0.316228	0.00849	2.6
2	1925	0.316226	0.01649	5.2

element. This slightly more expensive element results in solutions with improved accuracy for comparable computational cost.

Mesh Generation

Although existing mesh generators used in aerodynamics have proved very useful, they are not quite suitable for incompressible flow applications. Such grid generators usually output three-noded triangles, whereas finite element solvers for incompressible viscous flows require higher-order elements such as those shown in Fig. 1. Hence, a mesh generation code using the advancing front technique[5,6] was developed with the preceding characteristics.

Approximation Scheme

An augmented Lagrangian formulation incorporating the Uzawa algorithm and a Newton-based solver is used to solve the Navier-Stokes equations. The solver uses the two elements of Fig. 1.

The difference between the two elements lies in their convergence rate with grid refinement. The six-noded element has a linear convergence rate, in the H_1 norm for velocity and in the L_2 norm for pressure. This is a suboptimal performance given that a quadratic approximation is used for the velocity. The seven-noded triangle achieves an optimal quadratic rate of convergence for both velocity and pressure.

This higher rate of convergence is not simply of theoretical importance. Although it means that fewer degrees of freedom will be required to achieve a preset level of accuracy, in practice it often makes the difference between succeeding or failing to achieve a given accuracy.

Error Estimation

To our knowledge there exists no satisfactory adaptive strategy to solve the Navier-Stokes equations for incompressible flows. Error estimators used in compressible flows have proven inadequate here because the features of interest are different. Thus estimators specifically designed for incompressible flows are needed. A good estimator must have the following characteristics: 1) it must be rooted in the physics of the problem; 2) it must have a mathematical foundation; and 3) it must be convergent—meaning that as the number of degrees of freedom is increased the estimator should approach the true error.

A relatively simple error estimator based on postprocessing of the solution was introduced by Zhu and Zienkiewicz,[9] Zienkiewicz et al.,[10] and Ainsworth et al.[11] for elasticity and forming problems. For incompressible creeping-flow problems such an estimator can be derived from the variational formulation of Stokes flow. This consists in finding a velocity vector U with zero divergence that minimizes the dissipation energy functional:

$$J(U) = \int \mu(U_{i,j} + U_{j,i}) : (U_{i,j} + U_{j,i})\, dV = \int \sigma : \epsilon \, dV \quad (1)$$

If U_h is an approximate finite element solution and U_{ex} is the exact solution, the dissipation energy of the error

$$e = U_{ex} - U_h$$

can be evaluated by computing

$$J(e) = \int (\sigma_h - \sigma_{ex}) : (\epsilon_h - \epsilon_{ex})\, dV \quad (2)$$

Obviously, the exact solution will have zero error energy. The goal of the adaptation process will be to design a grid that will achieve a preset level of error. As can be seen, this estimator satisfies the first two criteria cited previously: it is rooted in the physics and has a rigourous mathematical foundation for creeping-flow problems. Unfortunately, the exact solution is not available in cases of practical interest. Hence, an approximation to $J(e)$ must be computed.

Note that whereas stresses and strains are continuous for classical solutions to the Navier-Stokes equations, the finite element scheme produces stresses and strains that are discontinuous across element faces. It is a simple matter to recover continuous stresses and strains from the finite element solution through a least-squares projection.[9] The estimator, using smoothed stresses σ_s and strain ϵ_s takes the form

$$J(e) = \int (\sigma_h - \sigma_s) : (\epsilon_h - \epsilon_s)\, dV \quad (3)$$

The error estimation for a given element is obtained by integrating Eq. (3) over the area of the element of interest. The global error is computed by integration over the whole domain. The energy norm of the error is finally obtained by taking

$$\|e_{tot}\|^2 = J(e)$$

This estimator was proven to be bounded and convergent for linear problems.[11] This provides a good starting point for developing an adaptive strategy for incompressible flows because it satisfies all of the requirements for creeping-flow problems. For nonlinear problems such as the Navier-Stokes equations, no general theory holds. Nevertheless, it should still prove useful since it is sensitive to high strain rates and shear stresses by construction. It should thus react favorably to boundary layers and stagnation points. Numerical simulations confirm this property.

Adaptive Strategy

The next key issue lies in how to adapt the mesh given the error estimate described previously. The remeshing strategy proceeds as follows:

1) generate an initial mesh
2) compute the finite element solution
3) compute error estimate
4) if (relative error < tolerance), then stop
   else compute grid function for mesh size; generate a new mesh according to grid function goto 2
   end if

Once a finite element solution has been obtained, the error on each element of the mesh is evaluated using Eq. (3). The total error and the norm of the solution are evaluated for the whole domain so that the relative error can be estimated:

$$\|e_{tot}\|^2 = \sum \|e_k\|^2$$

$$\eta = \|e_{tot}\| / \|U\|$$

The last step consists in computing the element size for the improved mesh such that elements are smaller in regions of large error and larger in regions where the solution is accurate. This is achieved by introducing the concept of optimal mesh. A grid is optimal if all finite elements of the mesh have the same average error e_{av}. Given a target relative error η_t, the total error can then be related to the yet unknown average element error:

$$n \|e_{av}\|^2 = \eta_t^2 \|U\|^2$$

thus

$$\|e_{av}\| = \eta_t \|U\| / n^{1/2}$$

Finally, element sizes for the next grid can be computed from the finite element asymptotic rate of convergence[11] which relates the error on element j, to some power k of the element size h:

$$\|e_j\| = Ch^k$$

which can also be written for the target error

$$\|e_{av}\| = C\delta^k$$

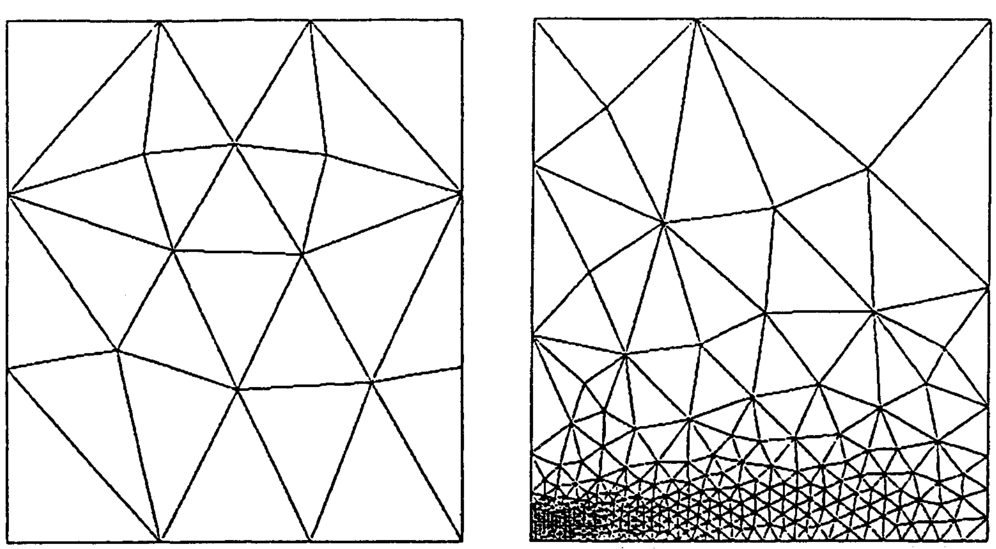

Fig. 4 Two-dimensional boundary layer: meshes.

Solving for the element size we get

$$\delta = \left[\frac{\eta_t \| U \|}{\| e \| n^{1/2}} \right]_h^{1/k}$$

This distribution is used as the grid function by the mesh generator to produce an improved grid.

Validation

Two simple boundary-layer problems with known analytical solutions are first solved to validate the strategy and assess its computational performance.

One-Dimensional Boundary Layer

For this problem, we assume that the solution is given by

$$u = 1 - \exp(-u_0 y/\nu)$$

$$v = 0$$

$$p = x$$

Values of the freestream velocity and viscosity determine the boundary-layer thickness.

The problem was first solved using the P2-P0 element with the adaptive strategy set to reduce the error by a factor of 4 at each cycle. Table 1 shows the convergence history. As can be seen, the error decreases with each cycle indicating that the adaptive strategy is convergent. The grids generated at each cycle are shown in Fig. 2. The strategy refines the grid throughout the domain in a roughly uniform manner. This is primarily due to the piecewise constant-pressure approximation of this element. Many elements are needed to accurately represent a linear pressure distribution.

Next, using the same element, the adaptation strategy was set to reach 1% relative error. Results are summarized in Table 2 and show that the grid is refined very rapidly with adaptation. The solution could not be obtained on the third grid because the number of grid points was so large that it prevented the global stiffness matrix from being held in core by the solver.

Tables 3 and 4 present analogous convergence histories for the Crouzeix-Raviart element. Figure 3 shows the sequence of

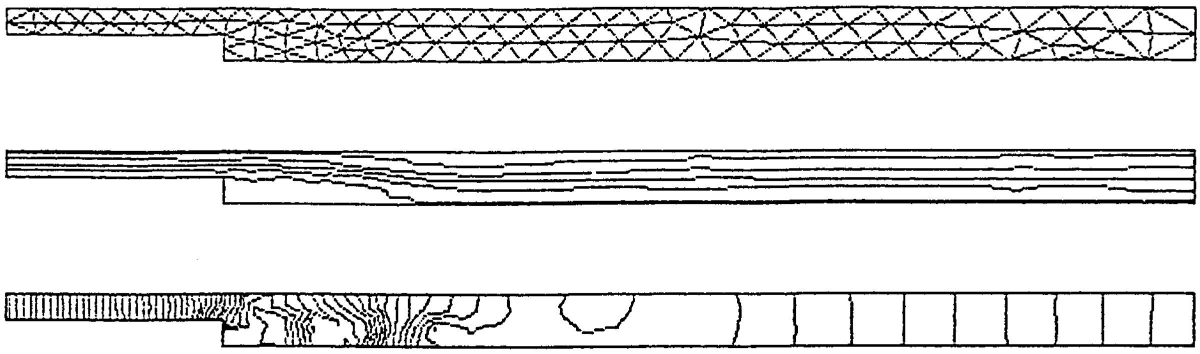

Fig. 5 Backward-facing step: initial mesh and solution.

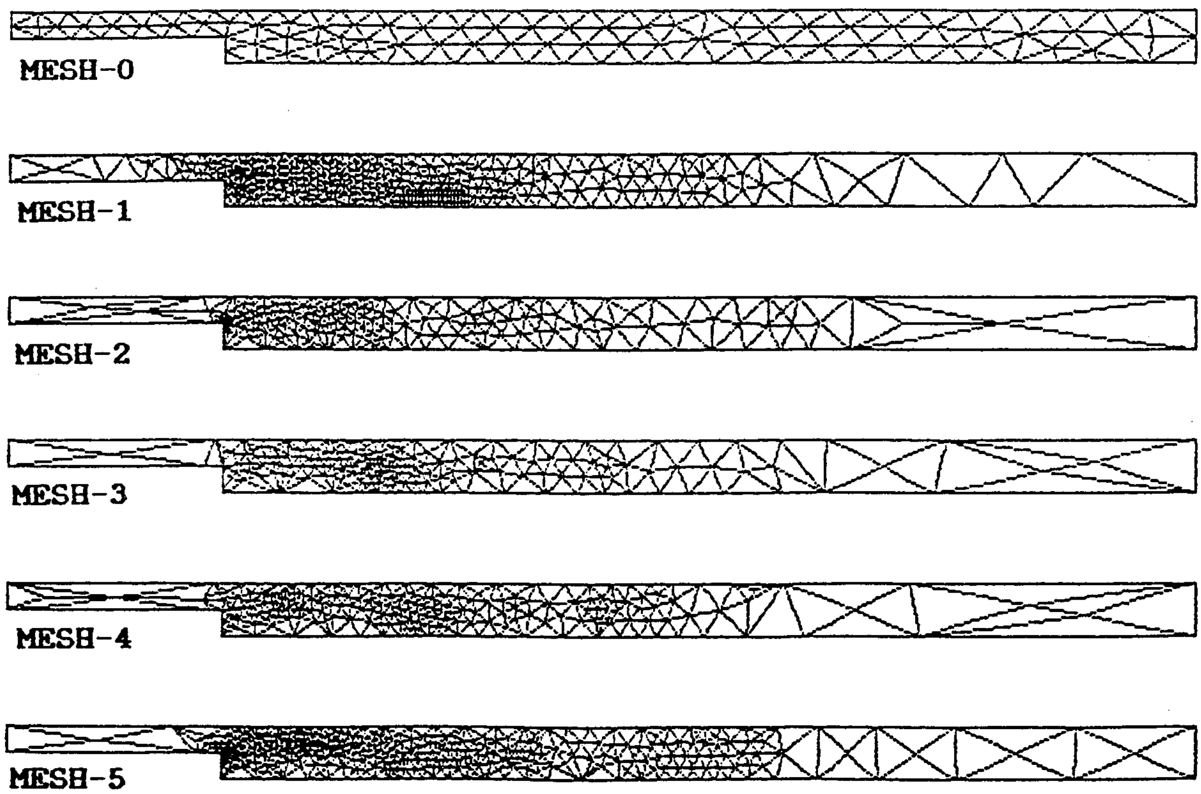

Fig. 6 Backward-facing step: meshes.

Table 3 One-dimensional boundary layer, Crouzeix-Raviart element, constant reduction of error

Mesh	Grid points	$\|U\|$	$\|e_{ex}\|$	Relative error, %
0	151	0.316228	0.02106	6.7
1	183	0.316226	0.01021	3.2
2	359	0.316226	0.00223	0.7
3	1125	0.316226	0.00063	0.2

Table 4 One-dimensional boundary layer, Crouzeix-Raviart element, target = 1% relative error

Mesh	Grid points	$\|U\|$	$\|e_{ex}\|$	Relative error, %
0	151	0.316228	0.02106	6.7
1	245	0.316226	0.00389	1.2
2	227	0.316226	0.00403	1.3
3	227	0.316226	0.00371	1.1

Table 5 Two-dimensional boundary layer, P2-P0 element target = 40% relative error

Mesh	Grid points	$\|U\|$	$\|e_{ex}\|$	Relative error, %
0	65	0.215084	0.36736	170
1	1550	0.219631	0.14989	68
2	1807	0.219631	0.12375	43
3	1909	0.219631	0.10946	39
4	2035	0.219631	0.10481	39

Table 6 Two-dimensional boundary layer, Crouzeix-Raviart element, target = 5% relative error

Mesh	Grid points	$\|U\|$	$\|e_{ex}\|$	Relative error, %
0	91	0.215084	0.11795	54.8
1	369	0.219629	0.02801	12.8
2	553	0.219631	0.01568	7.1
3	745	0.219631	0.01500	6.7

meshes generated. The grid is refined in the boundary layer and coarsened in the freestream. Comparison of Tables 2 and 4 presents eloquent results showing the advantage of using a higher-order discretization. More accurate results are obtained with nearly an order of magnitude fewer degrees of freedom resulting in a much higher computational efficiency.

Two-Dimensional Boundary Layer

A more realistic flow problem can be achieved by taking

$$U = 1 - \exp(-\xi)$$
$$P = x$$
$$\xi = y[U_0/(\nu x)]^{1/2}$$

The similarity variable ξ produces a velocity field closely resembling that of a boundary layer over a flat plate. This test case retains many nonlinear terms in the Navier-Stokes equations thus providing a more stringent test for the error estimator.

Tables 5 and 6 show the convergence histories for both elements. As before, the adaptation strategy is seen to be convergent and the Crouzeix-Raviart element remains far more accurate. Figure 4 shows the initial coarse grid and the one obtained after four cycles of adaptation. One can clearly see the smaller elements located in the upstream portion of the boundary layer. Note the slowly increasing element size consistent with the boundary-layer thickening.

Applications

In this section, we illustrate the usefulness of the proposed adaptive method by solving the classical problem of the flow over a backward-facing step[12] and comparing predictions using the Crouzeix-Raviart element with measurements.

Figure 5 shows the initial coarse grid (MESH-0). A parabolic velocity profile is specified at the inlet. No slip is enforced at the walls. Figures 6 and 7 show the meshes and pressure fields obtained at each adaptation cycle.

The flow has reverted to a fully developed parabolic state near the outlet (isobars are equally spaced vertical lines; see Fig. 7). Since the finite element approximation uses quadratic

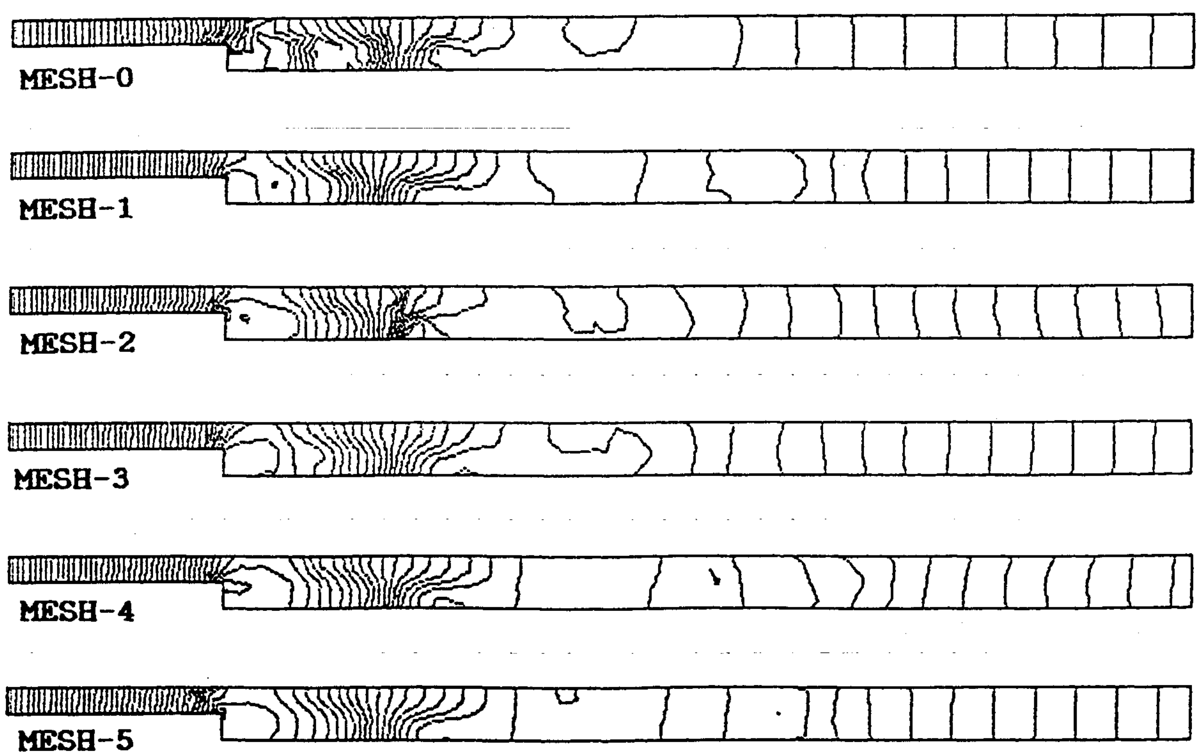

Fig. 7 Backward-facing step: pressure.

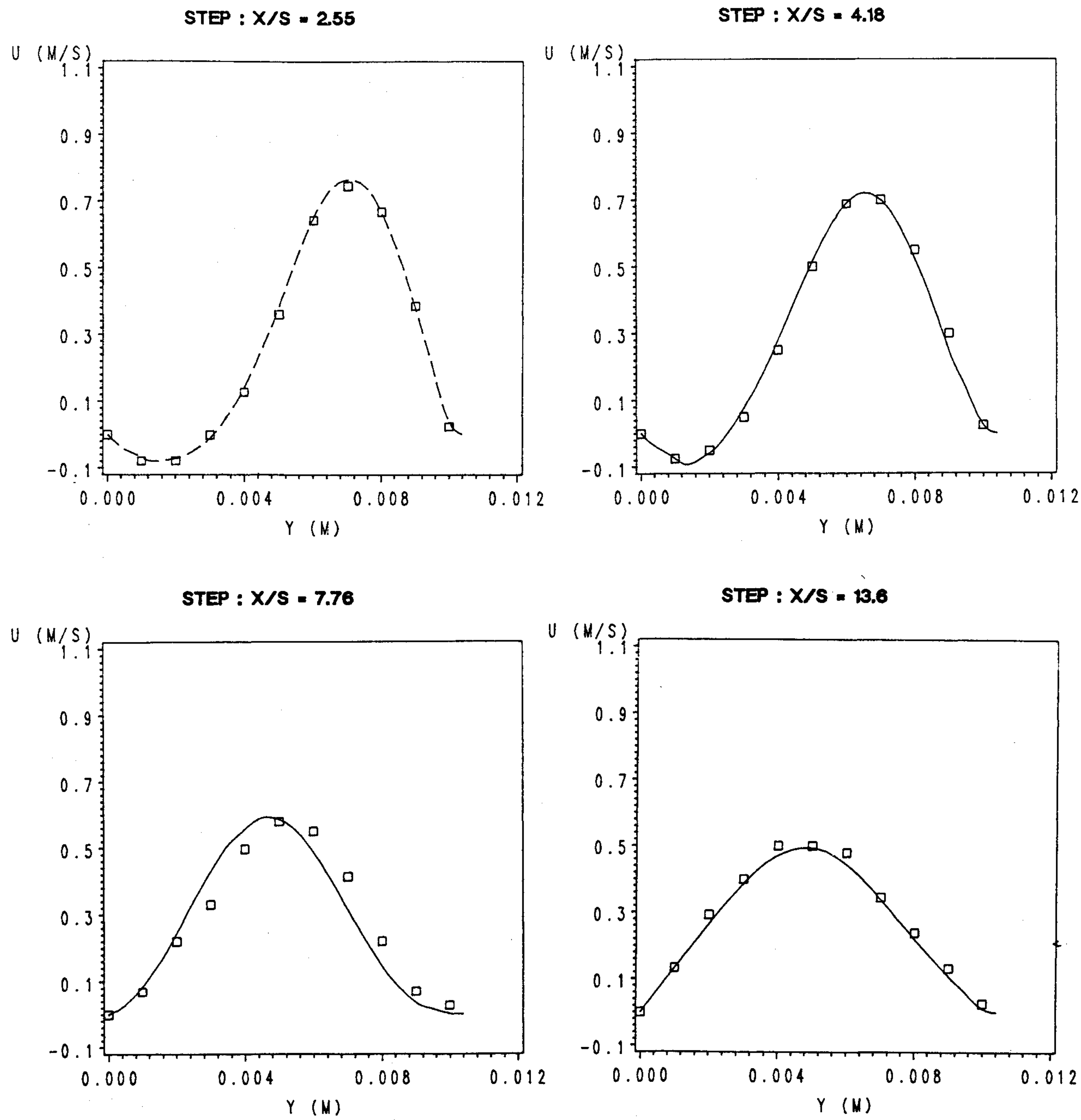

Fig. 8 Backward-facing step: comparison with experiments.

velocity, one element is sufficient to properly represent the solution near the inlet and outlet. Note that the adaptive strategy has produced such grids. Also note that the mesh was refined near the corner and the reattachment point.

For the second adaptation, the error estimator has detected that too many elements were generated in MESH-1 and has generated MESH-2 with fewer elements. However, the error estimate has decreased in spite of the reduction in the number of degrees of freedom indicating an improved allocation of degrees of freedom.

Finally, Fig. 8 shows a comparison between predictions and experimental measurements.[13] As can be seen the agreement is excellent everywhere except at station $X/S = 7.76$ just downstream of the reattachment point. Good predictions in this region of the flow are notoriously difficult to obtain due to the weak nature of the flow. A fine grid must be used in this region to obtain good results.[13] The present results were obtained by running the adapting strategy in a black box fashion. The mesh plots show a rather coarse grid in this region. This indicates that the error estimator does not perform sufficiently well in this region of the flow. In fact, near the reattachment point the stresses and strains are small due to the slow variation of the velocity field and the error estimator predicts a small error. However, pressure contour plots indicate that pressure varies more rapidly in this region. One way to improve the quality of the estimator could be to include pressure terms in Eq. (3).

Conclusions

An adaptive remeshing procedure has been presented for solving viscous incompressible flow problems.

The error estimator has been shown to be reliable and convergent by solving problems with known analytical solutions.

The estimator is sensitive to regions of high strain and can capture important features such as jets and shear layers.

Comparison with experimental measurements indicates that the estimator performs well except near reattachment points where velocities are low but pressure variations are rapid.

Inclusion of pressure terms in the estimator should improve its reliability near reattachment points.

The adaptive remeshing procedure has shown to be very robust. It can be used in a nearly black box fashion, requiring little or no intervention from the user.

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